

Attractors and The Holomorphic Anomaly

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ABSTRACT: Motivated by the recently proposed connection between $N = 2$ BPS black holes and topological strings, I study the attractor equations and their interplay with the holomorphic anomaly equation. The topological string partition function is interpreted as a wave-function obtained by quantizing the real cohomology of the Calabi-Yau. In this interpretation the apparent background dependence due to the holomorphic anomaly is caused by the choice of complex polarization. The black hole attractor equations express the moduli in terms of the electric and magnetic charges, and lead to a real polarization in which the background dependence disappears. Our analysis results in a generalized formula for the relation between the microscopic density of black hole states and topological strings valid for all backgrounds.

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1. Introduction

Recently an interesting connection has emerged between the counting of BPS black hole states and topological string theory. It was argued that the partition function of BPS black holes in Calabi-Yau compactifications of type II string theory is equal to the product of the partition sums of (anti-) topological strings [1]. As evidence it was shown that the corrected black hole entropy formula including its higher genus corrections takes the form of a Legendre transform of the sum of the free energies of the topological and anti-topological string. The relevant higher genus correction had earlier been determined in [2]. A central role in the correspondence is played by the black hole attractor equations [3, 4, 5], which express the values of the moduli fields at the horizon in terms of the electric and magnetic charges of the black hole. The proposed connection between BPS black holes and topological strings has been further studied in special cases with few charges, see for example [6, 7, 8, 9]. In this paper I will discuss the generic situation with arbitrary number of charges in the context of type IIB string theory on a compact Calabi-Yau.

According to [1] the mixed partition function associated with an extremal black hole with magnetic charge p^I and electric potential ϕ^I factorizes as

$$Z_{bh}(p, \phi) = \left| \exp \mathcal{F}_{top} \left(p + \frac{i}{2} \phi \right) \right|^2 \quad (1.1)$$

where \mathcal{F}_{top} denotes the free energy of the topological string on the Calabi-Yau space M . It has the perturbative expansion

$$\mathcal{F}_{top}(\lambda^{-1} X) = \sum_g \lambda^{2g-2} \mathcal{F}_g(X),$$

where λ is the string coupling constant and X^I are the periods of the holomorphic three form. In [1] the following identification was made,

$$\lambda^{-1} X^I = p^I + \frac{i}{2} \phi^I. \quad (1.2)$$

But there is a problem. The derivation of (1.1) as presented in [1] uses that \mathcal{F}_{top} is holomorphic in X^I . But it is well-known that the higher genus contributions \mathcal{F}_g have an anholomorphic dependence on the complex structure due to the holomorphic anomaly [10]. The problem therefore is to reconcile the proposed relation between black hole attractors and topological strings with the holomorphic anomaly.

Soon after the holomorphic anomaly was derived from the world sheet formulation [10], Witten interpreted the topological string partition as a wave function obtained by quantizing the space of three forms $H^3(M, \mathbb{R})$ on M in a complex polarization [11]. In this interpretation the holomorphic anomaly describes the behavior under an infinitesimal change of the polarization, analogous to the way the Knizhnik-Zamolodichikov equation appears in Chern-Simons theory. The wave function interpretation of the holomorphic anomaly has been further elaborated by Dijkgraaf, Vonk and the author [12].

In this paper I clarify the relation between the attractor equations and the holomorphic anomaly by applying the results of [11, 12]. I'll arrive at a somewhat modified proposal for the black hole partition function in terms of the topological string, one that does not suffer from the holomorphic anomaly and is background independent. The basic idea is as follows. Since the holomorphic anomaly is the result of a background dependent choice of complex polarization, one can remove it by using a background independent real polarization. The attractor equations suggest a real polarization, which instead of the complex moduli uses real variables related to the electric/magnetic charges and potentials.

The outline of this paper is as follows. I start with a review of special geometry in section 2. In section 3 I introduce the attractor equations and the entropy and

free energy of BPS black holes. The topological string partition function and the holomorphic anomaly are described in section 4. In section 5 I discuss the quantization of $H^3(M, \mathbb{R})$ in a complex and in a real polarization. The topological string is shown define a background independent state. The application to the black hole partition function of BPS black holes described in section 6, where I'll present a formula for the number of BPS states.

2. Special Geometry

The $N = 2$ supergravity theory corresponding to type IIB string theory compactified on a Calabi-Yau manifold M contains complex scalar fields X^I and vector fields A_μ^I with $I = 0, \dots, h_{2,1} = \dim H^{2,1}(M)$. The scalars X^I are identified with the periods of the holomorphic 3-form Ω on M along the non-intersecting A -cycles

$$\int_{A^I} \Omega = X^I. \quad (2.1)$$

The periods around the dual B -cycles with $\#(A^I, B_J) = \delta^I_J$

$$\int_{B^J} \Omega = \partial_I \mathcal{F}_0(X) \quad (2.2)$$

define a holomorphic function $\mathcal{F}_0(X)$ homogeneous in X with weight two. It represents the prepotential of the supergravity theory, and also equals the tree level free energy of the topological string. The first three derivatives of $\mathcal{F}_0(X)$ are denoted by

$$F_I \equiv \partial_I \mathcal{F}_0, \quad \tau_{IJ} \equiv \partial_I \partial_J \mathcal{F}_0, \quad C_{IJK} \equiv \partial_I \partial_J \partial_K \mathcal{F}_0. \quad (2.3)$$

The variables X^I parametrize the complex structure on the Calabi-Yau M .

The moduli space \mathcal{M} of complex structures is a special Kähler manifold with Kähler potential

$$K = -\log i \left(\overline{X}^I F_I - X^I \overline{F}_I \right) \quad (2.4)$$

Since overall scalings of Ω do not affect the complex structure, the X^I represent *projective* coordinates on \mathcal{M} . By taking the 3-form Ω to vary holomorphically with the complex structure, one can choose *local* coordinates t_i and \bar{t}_i on \mathcal{M} with $i = 1, \dots, h_{2,1}$, so that X^I and F_I are holomorphic in t_i . In this paper I'll use the local coordinates t_i and \bar{t}_i on \mathcal{M} as well as their projective counterparts X^I .

The freedom to scale Ω by a holomorphic function of the moduli leads to the Kähler gauge transformations

$$X^I(t) \rightarrow e^{f(t)} X^I(t), \quad \overline{X}^I(\bar{t}) \rightarrow e^{\bar{f}(\bar{t})} \overline{X}^I(\bar{t}) \quad (2.5)$$

and

$$K(t, \bar{t}) \rightarrow K(t, \bar{t}) - f(t) - \bar{f}(\bar{t}) \quad (2.6)$$

Meaningful equations are either invariant or covariant with respect to these transformations. Mathematically speaking X^I and F_I are holomorphic sections of a line bundle \mathcal{L} over \mathcal{M} defined by transition functions of the form (2.5). The three form Ω is also a section of \mathcal{L} . The line bundle \mathcal{L} comes equipped with a connection

$$\nabla_i = \partial_i + \partial_i K,$$

and its curvature is equal to the Kähler metric

$$G_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} K$$

on \mathcal{M} . An important role is played by holomorphic three point functions $C_{ijk}(t)$, which are obtained from the C_{IJK} in (2.3) by contracting each index with $\nabla_i X^I$. The Kähler potential, metric and three point function can be expressed as¹

$$\begin{aligned} e^{-K} &= i \int_M \bar{\Omega} \wedge \Omega, \\ e^{-K} G_{i\bar{j}} &= i \int_M \nabla_i \Omega \wedge \bar{\nabla}_{\bar{j}} \bar{\Omega} \\ C_{ijk} &= i \int_M \nabla_i \Omega \wedge D_j \nabla_k \Omega \end{aligned} \quad (2.7)$$

Here the covariant derivative D_i contains the usual christoffel connection Γ_{ij}^k as well as the term $\partial_i K$. Given a complex structure the third cohomology of the Calabi-Yau decomposes as

$$H^3(M) = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}.$$

The covariant derivative $\nabla_i \Omega$ and their complex conjugates form a basis of $H^{2,1} \oplus H^{1,2}$, which is isomorphic to the cotangent space of \mathcal{M} . Further derivatives of these three forms are therefore linearly related to the Ω and its first derivatives. The identities (2.7) imply that modulo exact terms one has

$$\bar{\partial}_{\bar{i}} \nabla_j \Omega = G_{i\bar{j}} \Omega, \quad D_i \nabla_j \Omega = -e^K C_{ij}^{\bar{k}} \bar{\nabla}_{\bar{k}} \bar{\Omega} \quad (2.8)$$

These relations will be useful in explaining the origin of the holomorphic anomaly. They further imply that the modified connection that includes $e^K C_{ij}^{\bar{k}}$ in its definition has zero curvature. This is a defining property of special Kähler geometry.

¹These relations can be verified using the Riemann bilinear identity

$$\int_M \alpha \wedge \beta = \sum_I \left[\int_{A^I} \alpha \int_{B_I} \beta - \int_{B_I} \alpha \int_{A^I} \beta \right] \quad \alpha, \beta \in H^3(M)$$

3. Black Hole Attractors

In this section I describe the attractor equations, and review the results of [1] on the free energy of BPS black holes and its connection with topological strings.

An extremal black hole with electric charges q_I and magnetic charge p^J corresponds in the type *IIB* theory to a three-brane wrapping the three-cycle

$$\mathcal{C} = q_I A^I - p^J B_J \quad (3.1)$$

on M . The BPS mass the black hole is expressed in terms of the period integral of Ω around \mathcal{C} ,

$$M_{BPS}^2 = e^K |\mathcal{Q}|^2, \quad (3.2)$$

with

$$\mathcal{Q} = \int_{\mathcal{C}} \Omega = q_I X^I - p^I F_I. \quad (3.3)$$

Here K denotes the Kähler potential (2.4). The linear combination (3.3) of the electric and magnetic charges q_I and p^J is known as the graviphoton charge. Notice that the BPS mass (3.2) is invariant under the combined Kähler transformations (2.5) and (2.6), but the graviphoton charge \mathcal{Q} itself is not.

The complex scalar fields X^I vary in general as a function of the radial coordinate r in the black hole geometry. For BPS black holes these fields reach special values at the black hole horizon characterized by the property that the expression (3.2) is minimized. This phenomena is known as the attractor mechanism. Equivalently the attractor values are obtained by minimizing the graviphoton charge while keeping fixed the Kähler potential. This leads to the condition

$$q_I - \bar{\tau}_{IJ} p^J = i\lambda^{-1} (F_I - \bar{\tau}_{IJ} X^J), \quad (3.4)$$

where λ^{-1} appears as a Lagrange multiplier. Its value is determined by contracting both sides with \bar{X}^I . This gives

$$\lambda^{-1} = e^K \overline{\mathcal{Q}} \quad (3.5)$$

The complex equation (3.4) is equivalent to two real equations

$$\text{Re}(\lambda^{-1} X^I) = p^I, \quad \text{Re}(\lambda^{-1} F_I) = q_I. \quad (3.6)$$

These are known as the attractor equations [4, 5]. The first equation is obtained by taking the imaginary part of (3.4). To find the second equation one first multiplies (3.4) by τ_{KI} before taking again the imaginary part. The quantity λ will in the next sections be identified with the coupling constant of the topological string. It behaves

under Kähler transformation as a section of \mathcal{L} , and hence the ratios $\lambda^{-1}X^I$ and $\lambda^{-1}F_I$ that appear in the attractor equations are invariant. Note further that the periods X^I and F_I as well as the charges p^I and q_I depend on the choice of canonical three cycles. Changing the choice of cycles transforms (X^I, F_I) and (p^I, q_I) by the same element of $Sp(2h_{2,1}+2, \mathbb{Z})$. The attractor equations (3.6) are therefore symplectically invariant.

The recent insight of [1] is that the attractor equations have a very natural thermodynamic interpretation. The Bekenstein-Hawking entropy S_{bh} for a BPS black hole with charges q_I and p^I is in leading order given by

$$S_{bh} = \frac{i}{2}|\lambda|^{-2} \left(X^I \overline{F}_I - \overline{X}^I F_I \right) \quad (3.7)$$

where the graviphoton charge X^I and F_I are evaluated at the attractor point. This expression can be written as

$$S_{bh} = \text{Im} \left(\lambda^{-2} X^I F_I \right) - 2 \text{Im} \left(\lambda^{-1} X^I \right) \text{Re} \left(\lambda^{-1} F_I \right) \quad (3.8)$$

The first term is equal to $2 \text{Im} \mathcal{F}_0(\lambda^{-1}X)$ due to the homogeneity of \mathcal{F}_0 . Now remember that the real part of $\lambda^{-1}X^I$ is set equal to the electric charge p^I by the first attractor equation. I'll denote its imaginary part by $\frac{1}{2}\phi^I$, so $X^I = p^I + \frac{i}{2}\phi^I$. The formula (3.8) for S_{bh} can then be recognized as the Legendre transform of

$$F_{bh}(p, \phi) = 2 \text{Im} \mathcal{F}_0 \left(p + \frac{i}{2}\phi \right). \quad (3.9)$$

One has

$$S_{bh}(p, q) = F_{bh}(p, \phi) - \phi^I \frac{\partial F_{bh}}{\partial \phi^I}(p, \phi) \quad (3.10)$$

where

$$q_I = \frac{\partial F_{bh}}{\partial \phi^I}(p, \phi) \quad (3.11)$$

is just the second attractor equation. The interpretation of $F_{bh}(p, \phi)$ is clear: it is the free energy associated with a black hole with magnetic charge p^I and electric potential ϕ^I . The surprising fact is that $F_{bh}(p, \phi)$ is given by the imaginary part of a holomorphic function of $p + \frac{i}{2}\phi$ equal to the genus zero free energy of the topological string. This observation could have been found many years ago, but only became apparent when it was recognized this relation persists beyond leading order.

Using the results of [2] it was shown in [1] that to all orders the free energy is equal to the sum of the free energies of the topological and anti-topological string. This was taken as evidence for that fact that the black hole partition function

associated with a mixed ensemble with electric potentials ϕ^I and fixed magnetic charges p^I

$$Z_{bh}(p, \phi) = \sum_q \Omega(p, q) e^{\phi \cdot q} \quad (3.12)$$

factorizes as

$$Z_{bh}(p, \phi) = \left| \Psi_{top}(p + \frac{i}{2}\phi) \right|^2. \quad (3.13)$$

Here Ψ_{top} denotes the topological string partition function. An interesting consequence is that the number of BPS states $\Omega(p, q)$ may then be expressed as (ignoring some factors of 2π)

$$\Omega(p, q) = \int d\chi e^{-i\pi\chi q} \Psi_{top}^*(p - \frac{1}{2}\chi) \Psi_{top}(p + \frac{1}{2}\chi) \quad (3.14)$$

This expression is just the Wigner function associated with $\Psi_{top}(p)$ when regarded as a wave function. The aim of this paper is to illuminate the wave function interpretation of the topological string partition function, and also to see what one can learn about topological string theory in general from this new perspective.

A useful ingredient in connecting the partition functions of topological strings and BPS black holes is the following description of the attractor equations in terms of $H^3(M, \mathbb{R})$. The 3-cycle \mathcal{C} in (3.1) corresponding to a BPS black hole with charges p^I and q_I is Poincare dual to a closed three form γ with real periods

$$\int_{A^I} \gamma = p^I, \quad \int_{B_J} \gamma = q_J. \quad (3.15)$$

Given a complex structure γ can be decomposed in the basis consisting of the holomorphic three form $\Omega \in H^{3,0}$, its covariant derivatives $\nabla_i \Omega \in H^{2,1}$ and their complex conjugates,

$$\gamma = \frac{1}{2} \left(\lambda^{-1} \Omega + x^i \nabla_i \Omega + \bar{x}^i \bar{\nabla}_i \bar{\Omega} + \bar{\lambda}^{-1} \bar{\Omega} \right). \quad (3.16)$$

By evaluating the periods of the r.h.s. and comparing the result with (3.15) one finds the following relation between λ and x^i and the periods

$$\text{Re} \left(\lambda^{-1} X^I + x^i \nabla_i X^I \right) = p^I, \quad \text{Re} \left(\lambda^{-1} F_I + x^i \nabla_i F_I \right) = q_I. \quad (3.17)$$

The attractor equations are recovered by putting $x^i = 0$, and hence they determine the complex structure for which γ has components only in $H^{3,0}$ and $H^{0,3}$. This observation is due to Moore [13]. In the following sections the parameters λ and x^i get an interpretation in the context of topological string theory: λ becomes identified with the perturbative coupling constant, while the x^i turn in to couplings to the physical operators.

4. The Holomorphic Anomaly

Topological string theory is defined perturbatively in terms of its world sheet description as a topological sigma model obtained by twisting an $N = 2$ superconformal field theory coupled to topological gravity [14]. I'll consider the B -model, which is distinguished from the A -model by the way the left and right moving $U(1)$ currents are used to twist the theory. In the B -model amplitudes depend on the complex structure moduli of the Calabi-Yau space, but are independent of the Kähler moduli.

The physical operators O_i of the B -model are in one to one correspondence with the $(2, 1)$ -forms on the target Calabi-Yau space M , which are in turn related to the complex structure deformations of M . Following standard string perturbation theory genus g amplitudes involving vertex operators O_i are expressed as integrated correlation functions: each operator is integrated over the Riemann surface Σ_g and subsequently the entire correlation function is integrated over the moduli space of genus g surfaces [10], see also [15]. By introducing couplings x^i for the operators O_i one defines a generating function of all genus g amplitudes

$$W_g(x; t, \bar{t}) = \left\langle \exp \sum_i x^i \int_{\Sigma_g} O_i \right\rangle_{t, \bar{t}} \quad (4.1)$$

The n -point amplitudes are found by successive differentiation w.r.t. x^i and putting $x^i = 0$ afterwards. The partition function of the topological string has the perturbative expansion

$$\Psi_{top}(x, \lambda; t, \bar{t}) = \lambda^{\frac{\chi}{24}-1} \exp \sum_g \lambda^{2g-2} W_g(\lambda x; t, \bar{t}) \quad (4.2)$$

where λ is the coupling constant and χ is the Euler number of M . For future convenience I inserted a factor λ in front of the x^i so that an n -point genus g amplitude is weighed with λ^{2g-2+n} . Only positive powers of λ appear because amplitudes with $2g - 2 + n \leq 0$ vanish. Due to the λ -dependent pre-factor Ψ_{top} is a section of $\mathcal{L}^{\frac{\chi}{24}-1}$.

To make the connection with the expression (3.14) the topological string partition function Ψ_{top} needs to be converted from a function of the couplings x^i and λ to a function of the charges p^I and/or potentials ϕ^I . An important clue comes from the ‘modified attractor relations’ (3.17). By themselves these are not yet sufficient, because they involve too many variables: both p^I and q_I occur and also the couplings x^i and λ appear together with their complex conjugates. As I'll explain in the following section, the correct way to convert Ψ_{top} to a function of the charge p^I is to regard Ψ_{top} as a wave function and to use (3.17) to convert it from a complex polarization to a real polarization. This same procedure will also remove the background dependence on the complex structure moduli t_i and \bar{t}_i .

The moduli dependence of Ψ_{top} is described by the holomorphic anomaly equations. There are two such equations, one describing the dependence on the anti-holomorphic moduli and the other on the holomorphic moduli. They read

$$\bar{\partial}_{\bar{i}}\Psi_{top} = \left[\frac{e^{2K}}{2} \bar{C}_{\bar{i}}^{jk} \frac{\partial^2}{\partial x^j \partial x^k} + G_{ij} x^j \frac{\partial}{\partial \lambda} \right] \Psi_{top} \quad (4.3)$$

$$\left[\nabla_i + \Gamma_{ij}^k x^j \frac{\partial}{\partial x^k} \right] \Psi_{top} = \left[\lambda^{-1} \frac{\partial}{\partial x^i} - \frac{1}{2} \partial_i \log |G| - \frac{1}{2} C_{ijk} x^j x^k \right] \Psi_{top} \quad (4.4)$$

where the connection is given by

$$\nabla_i = \partial_i + \partial_i K \left(\frac{h+1}{2} + x^j \frac{\partial}{\partial x^j} - \lambda \frac{\partial}{\partial \lambda} \right) \quad (4.5)$$

and $h = h_{2,1}$ and $|G| = \det G$. These equations were originally derived by world sheet techniques, but, as we will see in the next section, they have an interpretation that is quite independent of that in terms of the quantization of $H^3(M, \mathbb{R})$.

The holomorphic anomaly equations (4.3) and (4.4) differ slightly from those presented in [10]. Some small differences are due to the factor λ in front of x^i in (4.2). A more substantial difference is that the second anomaly equation given in [10] contains the genus one amplitude \mathcal{F}_1 instead of $\log |G|$ and also the coefficient in front of $\partial_i K$ in the connection is different. To arrive at (4.4) I eliminated \mathcal{F}_1 using its own anomaly equation [16]

$$\bar{\partial}_{\bar{i}} \partial_j \mathcal{F}_1 = \frac{e^{2K}}{2} \bar{C}_{\bar{i}}^{kl} C_{klj} - \left(\frac{\chi}{24} - 1 \right) G_{i\bar{j}} \quad (4.6)$$

This equation is solved by

$$\mathcal{F}_1 = \frac{1}{2} \log |G| + \left(\frac{h+1}{2} - \frac{\chi}{24} + 1 \right) K + f_1 + \bar{f}_1 \quad (4.7)$$

where f_1 is holomorphic in the moduli. In fact, (4.6) is contained in (4.3) by taking the linear term in x^i . The function f_1 is removed by multiplying Ψ_{top} with $\exp f_1$, which turns Ψ_{top} in to a section of $\mathcal{L}^{\frac{h+1}{2}}$ (times the square root of the holomorphic determinant bundle) instead of $\mathcal{L}^{\frac{\chi}{24}-1}$. By combining these steps the second holomorphic anomaly equation of [10] is turned in to (4.4). An important consequence of rewriting the second anomaly equation in this way is that it is now defined without any reference to the topological string partition function itself. Furthermore, both holomorphic anomaly equations are now linear, and therefore the set of solutions has become a linear space.

In the next section it will be shown that the linear space of solutions of the holomorphic anomaly equations can be identified with the space of wave functions obtained by quantizing $H^3(M, \mathbb{R})$ in a complex polarization. In particular, the topological string partition function will be identified with a state $|\Psi_{top}\rangle$. The second anomaly equation (4.4) is the conjugate of the first one (4.3) with respect to the norm

$$\langle \Psi_{top} | \Psi_{top} \rangle = \int d\mu_{x,\lambda} \exp \left[-e^{-K} G_{i\bar{j}} x^i \bar{x}^{\bar{j}} + e^{-K} (\bar{\lambda} \lambda)^{-1} \right] \left| \Psi_{top}(x, \lambda; t, \bar{t}) \right|^2 \quad (4.8)$$

where

$$d\mu_{x,\lambda} = d^2\lambda d^{2h}x |\lambda|^{-4} e^{\frac{h+1}{2}K} |G|^{\frac{1}{4}}$$

It is straightforward to verify that by differentiating the r.h.s. with respect to t_i or \bar{t}_i and using both holomorphic anomaly equations that at least formally this norm is independent of the background moduli. The significance of this fact will become clear in the following.

5. The Quantization of $H^3(M, \mathbb{R})$.

The third cohomology $H^3(M, \mathbb{R})$ has the structure of a phase space: it describes the classical solutions for the 7 dimensional action [17, 18], see also [19],

$$S = \int_{M \times \mathbb{R}} \gamma \wedge d\gamma,$$

where \mathbb{R} corresponds to the ‘quantization time’. The equations of motion imply that γ is time independent and closed. To quantize $H^3(M, \mathbb{R})$ one can use this action as a starting point, or equivalently, its natural symplectic form

$$Q_{symp} = \int_M d'\gamma \wedge' d'\gamma. \quad (5.1)$$

Here d' is an exterior derivative on the space of three forms, and hence the wedge product \wedge' on the r.h.s. of (5.1) is a combination of the one for three forms on M as well as the wedge product of $d'\gamma$ as a cotangent vector to $H^3(M, \mathbb{R})$. At this point there are two natural ways to proceed: either one parametrizes γ in terms of the couplings λ and x^i using the decomposition (3.16), or one uses its real periods p^I and q_J defined in (3.15). The first choice naturally leads to quantization of $H^3(M, \mathbb{R})$ in a complex polarization, also known as Kähler quantization, while the second choice gives a real polarization.

5.1 Kähler quantization.

To quantize $H^3(M, \mathbb{R})$ in the complex polarization one first chooses a complex structure on M . Next one decomposes $d'\gamma$ as in (3.16) in terms of infinitesimal variations of the couplings λ^{-1} and x^i . Explicitly,

$$d'\gamma = \frac{1}{2} \left(d\lambda^{-1}\Omega + dx^i \nabla_i \Omega + d\bar{x}^{\bar{i}} \bar{\nabla}_{\bar{i}} \bar{\Omega} + d\bar{\lambda}^{-1} \bar{\Omega} \right).$$

Inserting this decomposition in (5.1) and by making use of the relations (2.7) one finds for the symplectic form

$$Q_{symp} = ie^{-K} d\bar{\lambda}^{-1} \wedge d\lambda^{-1} + ie^{-K} G_{i\bar{j}} dx^i \wedge d\bar{x}^{\bar{j}}. \quad (5.2)$$

Generally, a symplectic form $Q_{ab} d\xi^a \wedge d\xi^b$ leads after quantization to the commutation relations $[\xi^a, \xi^b] = iQ^{ab}$, where Q^{ab} is the inverse of Q_{ab} . In this case

$$\left[\bar{\lambda}^{-1}, \lambda^{-1} \right] = e^K \quad \left[x^i, \bar{x}^{\bar{j}} \right] = e^K G^{i\bar{j}}. \quad (5.3)$$

These are the commutation relations of a set of creation and annihilation operators.

One can now associate a state $|\Psi_{top}\rangle$ with the topological string so that

$$\Psi_{top}(x, \lambda; t, \bar{t}) = \langle \Psi_{top} | x, \lambda \rangle. \quad (5.4)$$

where $|x, \lambda\rangle$ are the coherent eigenstates of x^i and λ . These states depend on t^i and \bar{t}^i , since λ^{-1} and x^i are defined w.r.t. a reference complex structure on M . In fact, all of the moduli dependence in the wave function Ψ_{top} is contained in these states; the state $|\Psi_{top}\rangle$ itself is background independent. To derive this result let me, following Witten [11], go back to the decomposition (3.16). Since γ itself is independent of the moduli, the variations of λ^{-1} and x^i should cancel those of the three forms Ω and $\nabla_i \Omega$. The latter are given in (2.8). In this way one finds

$$\bar{\partial}_{\bar{i}} \lambda^{-1} = -G_{\bar{i}j} x^j, \quad \bar{\partial}_{\bar{i}} x^k = e^K \bar{C}_{\bar{i}j}^k \bar{x}^{\bar{j}}$$

A change of \bar{t}_i therefore acts on the states as an infinitesimal Bogolyubov transformation, since it mixes creation and annihilation operators. After a little bit of puzzling one finds that the coherent states indeed satisfy the holomorphic anomaly equation

$$\bar{\partial}_{\bar{i}} |x, \lambda\rangle = \left[\frac{e^{2K}}{2} \bar{C}_{\bar{i}}^{jk} \frac{\partial^2}{\partial x^j \partial x^k} - G_{\bar{i}j} x^j \frac{\partial}{\partial \lambda^{-1}} \right] |x, \lambda\rangle \quad (5.5)$$

By commuting x^i and λ^{-1} through both sides of this equation one easily verifies that with this variation the state $|x, \lambda\rangle$ remains an eigenstate of these operators. This confirms that $|\Psi_{top}\rangle$ is indeed independent of the \bar{t} moduli.

In a similar way one can show that it is also independent of t^i . This fact also follows from the observation that the second holomorphic anomaly equation (4.4) is the conjugate of the first (4.3) with respect to the norm (4.8). This implies that the inner product of $|\Psi_{top}\rangle$ with any background independent state $|\Psi\rangle$ is also background independent. This can only be true if $|\Psi_{top}\rangle$ itself is independent of the background moduli.

The states $|x, \lambda\rangle$ are strictly speaking not part of the Hilbert space. This is due to the fact that λ^{-1} behaves like a creation operator instead of an annihilation operator, and hence its coherent states are not normalizable. This fact is also the cause of the upside-down gaussian integral over λ^{-1} in the expression (4.8) for the norm of $|\Psi\rangle_{top}$. To deal with this problem, let me also introduce the operator \mathcal{Q} corresponding to the graviphoton charge by writing $\bar{\lambda}^{-1} = e^K \mathcal{Q}$. Together with its conjugate it satisfies

$$[\mathcal{Q}, \bar{\mathcal{Q}}] = e^{-K}$$

The Hilbert space $\mathcal{H}_{H^3(M, \mathbb{R})}$ is spanned by the normalizable coherent eigenstates $|x, \mathcal{Q}\rangle$ of x^i and \mathcal{Q} . These states satisfy

$$\langle \bar{x}, \bar{\mathcal{Q}} | y, \mathcal{Q}' \rangle = \exp [e^{-K} G_{ij} \bar{x}^i y^j + e^K \bar{\mathcal{Q}} \mathcal{Q}'] \langle 0 | 0 \rangle \quad (5.6)$$

where $|0\rangle$ denotes the ground state. The coherent states $|x, \lambda\rangle$ can now be represented as formal integrals

$$|x, \lambda\rangle = \int d\mathcal{Q} \exp(-\lambda^{-1} \mathcal{Q}) |x, \mathcal{Q}\rangle \quad (5.7)$$

The norm (4.8) of $|\Psi_{top}\rangle$ can thus be reexpressed in terms of the overlaps with normalizable states $|x, \mathcal{Q}\rangle$. These obey the completeness relation

$$\mathbb{1} = \int d\mu_{x, \mathcal{Q}} \exp [-e^{-K} G_{ij} \bar{x}^i x^j - e^K \bar{\mathcal{Q}} \mathcal{Q}] |x, \mathcal{Q}\rangle \langle \bar{x}, \bar{\mathcal{Q}}| \quad (5.8)$$

where²

$$d\mu_{x, \mathcal{Q}} = d^2 \mathcal{Q} d^{2h} x e^{\frac{h-3}{2} K} |G|^{\frac{1}{4}}$$

The perturbative expansion of the overlaps $\langle \Psi_{top} | x, \mathcal{Q} \rangle$ is given by the Borel transform of the original expansion, and hence is possibly convergent. This is an indication that the state $|\Psi_{top}\rangle$ is not only background independent, but also normalizable.

²Consistency of (5.6) with (4.8) leads to $\langle 0 | 0 \rangle = \exp \frac{h+1}{2} K |G|^{\frac{1}{4}}$

5.2 Real polarization

The third cohomology $H^3(M, \mathbb{R})$ is defined in pure topological terms, and can be quantized in a background independent fashion by using the real periods p^I and q_J of γ instead of the complex couplings. Using the Riemann bilinear identity one writes the symplectic form Q_{sympl} in terms of the periods of the variation $d'\gamma$,

$$\int_{A^I} d'\gamma = dp^I, \quad \int_{B_J} d'\gamma = dq_J,$$

as

$$Q_{\text{sympl}} = \sum_I dp^I \wedge dq_I. \quad (5.9)$$

Here one recognizes the standard symplectic form on the phase space of q 's and p 's. The real periods q_I and p^J turn after quantization in to hermitean operators with the standard commutation relations

$$[p^I, q_J] = -i\delta^I_J \quad (5.10)$$

The periods p^I and q_J are related to the complex couplings λ and x^i through the equations (3.17). It is possible to verify that these indeed represent a canonical change of variables. The Hilbert space is spanned by the eigenstates of q_I or p^I . For definiteness, I'll work with the eigenstates $|p\rangle$ of the latter.

The wave functions obtained in the Kähler quantization can be converted to more standard wave functions in the real variables p^I or q_I . In particular, the state $|\Psi_{\text{top}}\rangle$ associated with the topological string can now be represented as a background independent wave function

$$\Psi_{\text{top}}(p) = \langle \Psi_{\text{top}} | p \rangle \quad (5.11)$$

Locally, the states $|p\rangle$ are indeed independent of the complex structure moduli. However, because p^I and q_I are defined w.r.t. a choice of 3-cycles, their eigenstates have non-trivial monodromy properties. Suppose a 3-cycle, say $C_{m,n} = m_I A^I - n^I B_I$, shrinks to zero size at a special locus in \mathcal{M} . By taking a closed path around its zero locus the other cycles pick up a monodromy

$$A^I \rightarrow A^I + n^I C_{m,n}, \quad B_I \rightarrow B_I + m_I C_{m,n}. \quad (5.12)$$

The periods p^I and q_I transform accordingly, that is

$$p^I \rightarrow p'^I = p^I + n^I (m_J p^J - n^J q_J), \quad q_I \rightarrow q'_I = q_I + m_I (m_J p^J - n^J q_J). \quad (5.13)$$

This makes clear that the effect of the monodromy is that the states $|p\rangle$ are changed by the unitary transformation

$$|p\rangle \rightarrow \exp \frac{i}{2}(m_I p^I - n^I q_I)^2 |p\rangle \quad (5.14)$$

Similar monodromies are picked up at all the loci at which a 3-cycle shrinks. Together they generate the modular group of the Calabi-Yau space M , which is a subgroup of $Sp(2h+2, \mathbb{Z})$. I will assume that the state $|\Psi_{top}\rangle$ is monodromy invariant. The wave function $\Psi_{top}(p)$, however, is in general not modular invariant but transforms under the modular group by canonical transformations. The significance of this observation will become clear in the next section.

5.3 The big phase space

To express the original topological partition function in terms of $\Psi_{top}(p)$, and vice versa, it is convenient to work in the ‘big phase space’ [12]. Instead of parametrizing the complex structure in terms of the local coordinates t_i and \bar{t}_i , one uses the projective coordinates X^I and its conjugate. Furthermore, the couplings x^i and λ are combined in to the variables

$$x^I = \lambda^{-1} X^I + x^i \nabla_i X^I. \quad (5.15)$$

These represent the coefficients of the decomposition $\gamma = \frac{1}{2}(x^I \omega_I + \bar{x}^I \bar{\omega}_I)$ in terms of the basis $\omega_I = \partial_I \Omega$ of $H^{3,0} \oplus H^{2,1}$. The relation between x^I and the real variables p^I and q_I is again determined by evaluating the periods of γ . This gives³

$$\text{Re}(x^I) = p^I \quad \text{Re}(\tau_{IJ} x^J) = q_I \quad (5.16)$$

These equations are equivalent to (3.17), and are solved explicitly by

$$x^I = -i (\text{Im} \tau^{-1})^{IJ} (q_J - \bar{\tau}_{JK} p^K). \quad (5.17)$$

The equations (5.16) and (5.17) should be regarded as operator identities. Therefore, one can take their expectation values or compute their matrix elements. For example, for the ground states $|0, \mathcal{Q}\rangle$ this leads to the interesting property that the

³The period integrals of the forms ω_I are

$$\int_{A^I} \omega_J = \delta^I_J \quad \int_{B_I} \omega_J = \tau_{IJ}.$$

normalized expectation values of the charge operators p^I and q_I obey the attractor equations

$$\langle p^I \rangle = \text{Re}(\lambda^{-1} X^I), \quad \langle q^I \rangle = \text{Re}(\lambda^{-1} F_I), \quad (5.18)$$

where $\lambda^{-1} = e^K \overline{\mathcal{Q}}$. This follows from the fact that the states $|0, \mathcal{Q}\rangle$ are annihilated by x^i and hence the expectation value of x^I is equal to $\lambda^{-1} X^I$.

From the canonical commutators (5.10) and (5.17) it is clear that x^I and its conjugate satisfy

$$[x^I, \overline{x}^J] = 2 (\text{Im}\tau^{-1})^{IJ}.$$

This commutator is equivalent to those given in (5.3) for the couplings x^i and λ^{-1} . The states $|x, \lambda\rangle$ can thus be identified with the coherent eigenstates $|x\rangle$ of x^I . The completeness relation for these states is

$$\mathbb{1} = \sqrt{|\text{Im}\tau|} \int dx d\overline{x} e^{-\frac{1}{2}x^I \overline{x}_I} |x\rangle \langle \overline{x}|, \quad (5.19)$$

where $|\text{Im}\tau|$ denotes the determinant of $\text{Im}\tau_{IJ}$. Here and in the following indices are raised and lowered using $\text{Im}\tau_{IJ}$ as the metric on the big phase space.

To be able to convert wave functions from the complex to the real polarization one needs the overlaps $\langle p|x\rangle$. By requiring that the relations (5.16) hold as operator identities when inserted in between the two states $\langle p|$ and $|x\rangle$ one finds [12]

$$\langle p|x\rangle = \exp \frac{i}{2} (p^I \overline{\tau}_{IJ} p^J - p^I x_I + \frac{1}{2} x^I x_I). \quad (5.20)$$

It thus follows that the original partition function can be expressed in terms of $\Psi_{top}(p)$ by

$$\Psi_{top}(x) = \int dp \Psi_{top}(p) \exp \frac{i}{2} (p^I \overline{\tau}_{IJ} p^J - p^I x_I + \frac{1}{2} x^I x_I). \quad (5.21)$$

The inverse relation expressing $\Psi_{top}(p)$ in terms of $\Psi_{top}(x)$ is easily obtained with the help of the completeness relation (5.19). Finally, from the expression (5.21) it is almost manifest that the partition function satisfies the holomorphic anomaly equations. In the big phase space these takes the form

$$\begin{aligned} \overline{\partial}_I \Psi_{top} &= \frac{1}{2} \overline{C}_I{}^{JK} \frac{\partial^2}{\partial x^J \partial x^K} \Psi_{top}, \\ \left[\partial_I + C_{IJ}^K x^J \frac{\partial}{\partial x^K} \right] \Psi_{top} &= \left[-\frac{1}{2} \partial_I \log |\text{Im}\tau| - \frac{1}{2} C_{IJK} x^J x^K \right] \Psi_{top} \end{aligned}$$

For more details on the ‘big phase space version’ of the holomorphic anomaly equation, see [12].

6. BPS States and Topological Strings

Much of the material presented in the previous sections is in some form or another contained in or scattered throughout the literature. I have gone through it in detail to make the paper self-contained, fill in gaps and to highlight the interplay between the attractor equations and the holomorphic anomaly. But now that the stage is set, let me return to discussing the proposed relation between BPS black holes and topological strings.

Topological string theory has at present only a perturbative formulation. In this respect the correspondence with the BPS black holes is particularly interesting because it may provide non-perturbative formulation. It also strongly suggests that such a formulation involves both topological and anti-topological strings. In this section I take the perspective that the counting of BPS black hole states serves as a *definition* of non-perturbative topological string theory.

Let $\Omega(p, q)$ be the number of black hole states counted with weight $(-1)^F$. One can define an operator in the Hilbert space obtained by quantizing $H^3(M, \mathbb{R})$ by

$$\hat{\Omega} = \sum_{p,q} \int d\chi e^{i\chi q} |p - \tfrac{1}{2}\chi\rangle \Omega(p, q) \langle p + \tfrac{1}{2}\chi| \quad (6.1)$$

where $|p \pm \tfrac{1}{2}\chi\rangle$ are the states obtained in the real polarization. The relation (6.1) can be inverted to

$$\Omega(p, q) = \int d\chi e^{-i\chi q} \langle p - \tfrac{1}{2}\chi | \hat{\Omega} | p + \tfrac{1}{2}\chi \rangle. \quad (6.2)$$

This equation holds independent of the connection with topological strings. The expression (3.14) described in section 3 gives the relation between the number of BPS states $\Omega(p, q)$ and the topological string in terms of the real polarization. The formula (3.14) implies that the operator $\hat{\Omega}$ factorizes as

$$\hat{\Omega} = |\Psi_{top}\rangle \langle \Psi_{top}| \quad (6.3)$$

In other words, $\hat{\Omega}$ is (proportional to) a projection operator on the state $|\Psi_{top}\rangle$. In fact, below I will argue that this relation is presumably only approximately true. Namely, the r.h.s. of (6.3) should probably be generalized to include a sum over states, so that $\hat{\Omega}$ behaves more like a density matrix. Independent evidence for this fact is given in [6] and [20].

The results of the previous section can now be used to convert the expression (6.2) from the real to the complex polarization by writing it in terms of the coherent

states $|x\rangle$ using the overlap (5.20). The key identity is

$$\int d\chi e^{-iq\chi} \langle x|p + \frac{1}{2}\chi\rangle \langle p - \frac{1}{2}\chi|\bar{x}\rangle = \frac{e^{-\frac{1}{2}\|x-x_{p,q}\|^2}}{\sqrt{|\text{Im}\tau|}} \quad (6.4)$$

where $x_{p,q}$ are the solutions to (5.16) given in (5.17), and the norm in the exponential is defined by

$$\|x - x_{p,q}\|^2 \equiv \text{Im}\tau_{IJ} (x^I - x_{p,q}^I) (\bar{x}^J - \bar{x}_{p,q}^J) \quad (6.5)$$

This norm is not positive definite, however, since $\text{Im}\tau_{IJ}$ has one negative eigenvalue corresponding to $x^I \sim X^I$. This problem can be solved along the lines discussed in section 4.1, but for simplicity I'll ignore this issue in the remainder.

Assuming that (6.3) holds the matrix element $\langle \bar{x}|\hat{O}|x\rangle$ factorizes as a product of $\Psi_{top}(x)$ and its conjugate. Using (6.1) and (6.4) one thus finds

$$\Psi_{top}^*(\bar{x})\Psi_{top}(x) = \sum_{p,q} \Omega(p,q) \frac{e^{-\frac{1}{2}\|x-x_{p,q}\|^2}}{\sqrt{|\text{Im}\tau|}} \quad (6.6)$$

The r.h.s. defines a new partition function of BPS black holes corresponding to a particular ensemble with a statistical weight given by gaussian factor. This ensemble is apparently the natural one from the point of view of the topological string. It would be interesting to know what is special about this ensemble.

The expression for the $\Omega(p,q)$ in terms of the topological partition function in complex polarization is obtained by inverting the relation (6.6). Alternatively, one can start from (3.14) and use the completeness relations (5.19). Both methods yield the following result

$$\Omega(p,q) = \sqrt{|\text{Im}\tau|} \int dx d\bar{x} e^{-\frac{1}{2}\|x\|^2} \Psi_{top}^*(\bar{x}_{p,q} - \bar{x}) \Psi_{top}(x_{p,q} + x) \quad (6.7)$$

The reversal of the sign in front of x is necessary so that the gaussian integration is (mostly) convergent. Again to get a completely convergent result one has to change to the normalizable polarization in terms of \mathcal{Q} .

The formula (6.7) is one of the main results of this paper. It is written in a short hand notation to make it readable. But hidden in the wave functions $\Psi(x)$ is still the dependence on the background moduli. The full expression is nevertheless independent of the choice of background moduli. Hence, one is free to choose the background at special points. For example, by sending the moduli to ‘infinity’ in such a way that $\text{Im}\tau_{IJ} \rightarrow \infty$ one recovers the expression (3.14) as a special case. Alternatively, one can for instance choose the moduli to be at the attractor

point. In that case the value of $x_{p,q}$ is equal to the solutions $X_{p,q}$ of the attractor equations. By expanding Ψ_{top} and its conjugate around the background one can derive a perturbative expressions for $\Omega(p, q)$ in terms of the correlation functions of the topological string. I'll leave it for future work to analyze the resulting expressions in detail.

Let me end with a few comments and open problems. First of all, from the result (6.7) it is not immediately clear that $\Omega(p, q)$ are integers. Indeed, even the integral nature of the charges p^I and q_J has hardly been used in this paper. This brings us to a final comment regarding the factorized form (6.3) of $\hat{\Omega}$. Without claiming to have a complete understanding of this point, I believe that it is unlikely that this identity holds exactly. This has to do with the following important property that is expected to hold for the number of BPS states $\Omega(p, q)$.

Since $\Omega(p, q)$ is integer it should indeed be independent of small changes of the background complex structure. Now, one can perform a monodromy that changes the choice of basis of three cycles, and hence also the definition of the charges by the monodromy transformations (5.13). From this one concludes that $\Omega(p, q)$ should be monodromy invariant: that is

$$\Omega(p, q) = \Omega(p', q'), \quad (6.8)$$

where p' and q' are given in (5.13). This is clearly a non-trivial property, comparable to the restriction of modular invariance for 2d conformal field theory. It implies that the operator $\hat{\Omega}$ commutes with the generators (5.14) of the monodromy transformation on the states. Now, if $\hat{\Omega}$ indeed factorizes as in (6.3) this would mean that the state $|\Psi_{top}\rangle$ itself is invariant. In other words

$$e^{\frac{i}{2}(m \cdot p - n \cdot q)^2} |\Psi_{top}\rangle = |\Psi_{top}\rangle \quad (6.9)$$

for all cycles $m_I A^I - n^I B_I$ that can shrink to zero. This is analogous to demanding modular invariance of a conformal block, which is generally only possible for very special CFT's. Clearly this point needs further investigation.

Another more intuitive reason why the factorized form (6.3) is not likely to hold exactly is that it would mean that the topological string by itself has a non-perturbative definition. But the correspondence with the BPS black holes seems to suggest that the non-perturbative definition involves both topological as well as anti-topological strings. The formula (6.7) should then be modified so that there is a sum over states on the r.h.s. So one gets a density matrix formulation of the non-perturbative theory defined by the numbers $\Omega(p, q)$. This point of view will be further elaborated in forthcoming work [21].

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